Optimal Design of Batch-Storage Network With Financial Transactions and Cash Flows

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DOI 10.1002/aic.10273
Published online in Wiley InterScience (www.interscience.wiley.com).

An integrated analysis of production and financing decisions is presented. A model is constructed in which a cash storage unit is installed to manage the cash flows associated with production activities, such as raw material procurement, process operating setup, inventory holding costs and finished product sales. Temporary financial investments are allowed to increase profit. The production plant is modeled by the batch-storage network model with recycle streams. The objective function of the optimization is minimizing the opportunity costs of annualized capital investment and cash/material inventory minus the benefit to stockholders. The major constraints of the optimization are that the material and cash storage units must not be depleted. A production and inventory analysis formulation, the periodic square wave (PSW) model, provides useful expressions for the upper/lower bounds and average levels of the cash and material inventory holdups. The expressions for the Kuhn-Tucker conditions of the optimization problem are reduced to a subproblem and analytical lot sizing equations. This subproblem is then decomposed into two separable concave minimization network flow problems whose solutions yield the average material and cash flow rates through the networks. The production and financial transaction lot sizes and startup times can be determined by analytical expressions after the average flow rates are already known. It is shown that, when financial factors are taken into consideration, the optimal production lot and storage sizes are smaller than is the case when such factors are not considered. An illustrative example is presented to demonstrate the potential of this approach. *AIChE J,* 50: 2849–2865, 2004
Keywords: optimal, lot-size, batch- storage network, financial, cash flow

Introduction

Most production planning and scheduling models developed to date in process system engineering endeavor to identify a plan or schedule that minimize the overall cost while satisfying production capacity and demand constraints. A key assumption of these models is that an unlimited amount of cash is available. In practice, however, cash is usually the scarcest resource and cash availability is an important factor influencing the feasibility of a production plan or schedule. It is commonplace for a planned production to be unrealizable for a period of time due to a lack of cash to cover the production costs, resulting in other resources being under-utilized during that period. Then, when the cash does become available, overproduction must be conducted to fill backorders. These inefficiencies can cause substantial loss of profit. In fact, every aspect of production involves financial transactions and cash flows. Manufacturers

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purchase raw materials for production purposes, creating accounts payable owed to suppliers. The actual disbursement of cash occurs when the payment medium used to pay for the purchase, such as a check, is redeemed through the bank system. Raw materials are converted into finished products by consuming operational utilities that incur costs associated with their purchase or production. The finished product inventory is converted into accounts receivable as customers make purchases on credit. Receivables are then collected from customers remitting payment to the company. Cash is received when the payment medium, such as a check, is redeemed through the bank system. In the mean time, the company must pay taxes, salaries, and disburse loans. To prevent temporary cash shortages in some circumstances, new loans must be arranged. If there is excess cash, temporary investment in marketable securities should be considered to increase income. If operating cash flows are not well managed, seemingly profitable firms may experience financial strains that could potentially lead to bankruptcy. For example, if too many resources are tied up in inventory or accounts receivable, then even a profitable company may not be able to pay its bills. Therefore, it is essential to consider cash flow when making production planning and scheduling decisions. A successful firm manages its operations so as to optimize both profit and cash flow.

In traditional plant design, the three aspects of decisions - investment, finance and operations - are analyzed in a sequential fashion with little effort devoted to incorporating the mutual complex interactions among these three aspects. This approach has the inherent attraction that it is compatible with the typical corporate management structure wherein different decision making departments can have more flexibility. Moreover, the segregation of the three components greatly simplifies the design problem. However, the interactions among the decisions made regarding the investment, finance and operations aspects may significantly impact the overall business structure and ignoring such interactions may result in a perpetual burden in the management of funds. It is likely that the investment plan devised considering all financial constraints would differ substantially from the plan generated by the traditional fragmented approach.

Kirca and Koksalan (1996) presented an integrated financial and production planning model that was developed, based on a multiperiod mixed integer linear programming model. Their model considered labor costs, severe interest rate inflation, collection drift of accounts receivable, bank credits and cash flow, as well as demands, machine capacities and material requirements. Their approach is very general, but may involve a prohibitively high computational burden when applied to large-sized manufacturing companies. Majumdar and Chattopadhyay (1999) considered the problem of designing a plant with expanded electricity generation capacity including financial constraints. In the field of chemical engineering, little work has been done on incorporating the financial and cash flow aspects into production management. Badell and Puigjaner (2001) suggested a software design methodology for enterprise resource management that included financial transactions and cash flow.

Yi and Reklaitis (2000) developed a novel production and inventory analysis method called the periodic square wave (PSW) method, and used it to determine the optimal design of a parallel batch-storage system. They subsequently extended the PSW formalism to model the more complicated plant structure of a sequential multistage batch-storage network (Yi and Reklaitis, 2002). In another study (Yi and Reklaitis, 2003), the same authors suggested a nonsequential network structure that can deal with recycled material flows in a plant site. In this study, we extend the batch-storage network model suggested by Yi and Reklaitis (2003) to include both the cash storage and the financial transactions required to support the production activities. In the proposed model, all production activities are accompanied by financial transactions in which the appropriate amount of cash is withdrawn from the cash storage to pay for the costs. Cash is inputted to the storage after delivery of the finished product to consumers. The cash inventory should be managed so as to ensure that it is not depleted. The objective function of the optimization is minimizing the opportunity costs of annualized capital investment and cash/material inventory minus the benefit to stockholders.

Our theory is based on the following major assumptions:

(1) All production and financial operations are periodical.

(2) The material and cash flows are balanced in the long run.

(3) The depletion of cash and material inventories is not allowed.

(4) The lot size of the dividend to stockholders is constant.

More detailed assumptions are necessary to define the batch-storage network for production; these assumptions have been summarized previously by Yi and Reklaitis (2003).

Definition of variables and parameters

We use the plant structure introduced by Yi and Reklaitis (2003); below we give a brief overview of the definitions of variables and some equations required in this study. A chemical plant, which converts raw materials into final products through multiple physicochemical processing steps, is composed of a set of storage units (J) and a set of batch processes (I) as shown in Figure 1a. Each process requires multiple feedstock materials of fixed composition (f_i^j), and produces multiple products with fixed product yield (g_i^j), as shown in Figure 1b. If there is no material flow between a storage unit and a process unit, the corresponding feedstock composition or product yield value is zero. Each storage unit is dedicated to one material and, therefore, the storage index j is also the material index. As shown in Figure 1c, each storage unit is involved in four types of material movement: purchasing from suppliers ($k \in K(j)$), shipping to consumers ($m \in M(j)$), feeding to processes and production from processes. Note that the sets of suppliers $K(j)$ and consumers $M(j)$ are storage dependent. The material flow from process to storage unit (or from storage unit to process) is represented by the periodic square wave (PSW) model, as shown in Figure 5 in Yi and Reklaitis (2002). In the PSW model, material flow is represented in terms of four variables: batch size, cycle time ω_i, storage operation time fraction x_i^{in} (or x_i^{out}), and startup time t_i^{in} (or t_i^{out}). The storage operation time fraction x_i^{in} (or x_i^{out}) is defined as the time required for material movement to (or from) the process divided by the cycle time. The startup time t_i^{in} (or t_i^{out}) is the first time at which the first batch is fed into (or discharged from) the process. We assume that the operations feeding feedstock to the process (or the operations discharging the product from the process) occur simultaneously, and that their storage operation time fractions are the same. That is, the

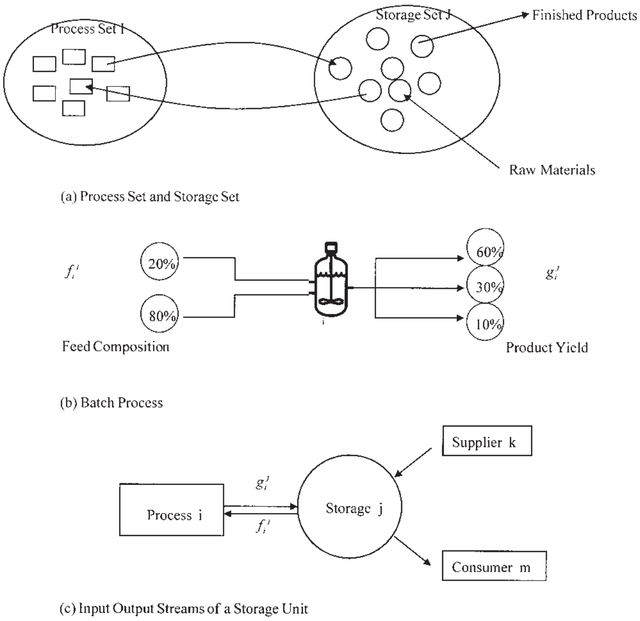

Figure 1. General structure of batch-storage network.

(a) process set and storage Set, (b) batch process, (c) input output streams of a storage unit.

superscript j is not necessary to discriminate the storage units in x_i^{in} (or x_i^{out}) and t_i^{in} (or t_i^{out}). The material flow of purchased raw material is represented by the order size B_k^j, cycle time ω_k^j, storage operation time fraction x_k^j, and startup time t_k^j. All storage operation time fractions will be considered as parameters, whereas the others will be the design variables used in this study. The material flow of finished product sales is represented by B_m^j, ω_m^j, x_m^j and t_m^j in the same way. The arbitrary periodic function of the finished product demand forecast can be represented by a sum of periodic square wave functions with known values of B_m^j, ω_m^j, x_m^j and t_m^j (Yi and Reklaitis 2000).

Let D_i be the average material flow rate through process i, which can be expressed as batch size B_i divided by cycle time ω_i. The average material flows of raw material purchase from suppliers and finished product shipping to consumers are denoted by D_k^j and D_m^j, respectively, where $D_k^j = B_k^j/\omega_k^j$ and $D_m^j = B_m^j/\omega_m^j$. The initial inventory of storage j is denoted by $V^j(0)$ and the inventory holdup of storage j at time t is denoted by $V_j(t)$.

The purchasing setup cost of raw material j is denoted by A_k^j \$/order, and the setup cost of process i is denoted by A_i \$/batch. The annual inventory holding cost of storage j is denoted by H^j \$/L/year. The inventory holding cost is further segregated into the inventory operating cost (h^j), and the opportunity cost of inventory holding (γ^j), that is, $H^j = h^j + \gamma^j$. We assume that capital cost is proportional to process capacity in order to obtain an analytical solution. Suppose that a_k^j (\$/year/L) is the annual capital cost of the purchasing facility for raw material j, a_i (\$/year/L) is the annual capital cost of process i, and b^j (\$/year/L) is the annual capital cost of storage unit j. In addition, assume that the raw material cost is proportional to the quantity and purchase price of raw material j from supplier k is P_k^j (\$/L). The sales price of finished product j to consumer m is S_m^j (\$/L). Appendix A summarizes the necessary equations derived in our previous work (Yi and Reklaitis, 2003).

Suppose that there exists a cash storage unit of the type shown in Figure 2 that, through financial transactions, operates the chemical plant composed of batch process set I and material storage unit set J, as depicted in Figure 1. Let set N with subscript n represent the set of temporary financial investments in marketable securities, and set O with subscript o represent the set of stockholders. Corporation income tax is usually proportional to net profit and is, thus, considered as a payment to a fictitious stockholder without loss of generality. Sales tax, which is usually proportional to sales revenue, is collected from customers when finished products are delivered to them and is paid to the Internal Revenue Service (IRS) yearly. In chemical companies, total labor cost is usually proportional to total sales revenue. We ignore the cash flow of labor costs in this study because it is treated in the same way as sales tax. Note that the setup cost usually includes the operating labor cost. The incoming cash flows into the cash storage unit are:

CF1) Collection of account receivable after collection drifting time Δt_m^j from shipping of the finished product to consumer m. (Sales tax is included.)

CF2) Return of temporary financial investment n at interest rate κ_n (\$/\$/year) after investment period Δt_n.

The outgoing cash flows from the cash storage unit are:

CF3) Disbursement of account payable after disbursement drifting time $\Delta t' k$ for raw material purchase from supplier k.

CF4) Temporary financial investment at interest rate κ_n (\$/\$/year) for investment period Δt_n.

CF5) Financial transaction of purchase setup cost.

CF6) Financial transaction of investment setup cost.

CF7) Financial transaction of processing setup cost.

CF8) Inventory operating cost.

CF9) Dividend to stockholders (can include corporation income tax).

CF10) Sales tax payment to IRS with tax rate ς (\$/\$). (Labor cost can be treated as the same way.)

In this work we do not consider the case of the corporation taking bank loans to top up their cash reserves because, once any initial cash shortage has been addressed, it should be unnecessary to take further loans, and such loans consume the benefits to stockholders'. We assume that the temporary finan-

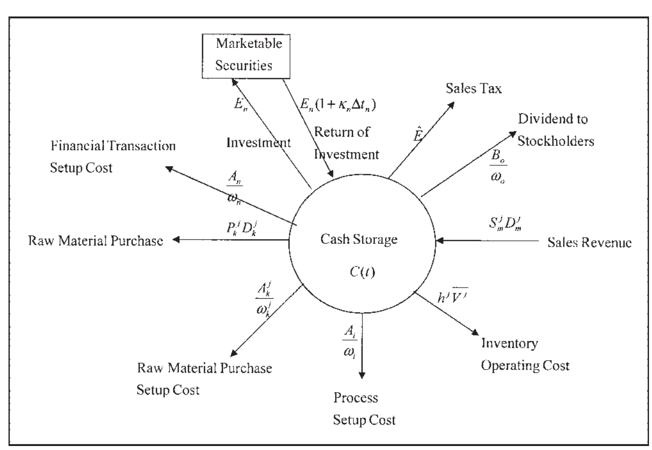

Figure 2. Cash Storage and financial transactions.

Table 1. Mathematical Properties of PSW Flows

Average of first type PSE flow	$\overline{PSW}(t;\, D,\, \omega,\, t',\, x) = D(t - t') + 0.5(1 - x)D\omega$
Average of second type PSW flow	$\overline{PSW'}(t; B, \omega, t', x) = \frac{B}{\omega}(t - t') + 0.5(1 - x)B$
Upper bound of first type PSW flow	$PSW(t; D, \omega, t', x) = D(t - t') + (1 - x)D\omega$
Upper bound of second type PSW flow	$\overline{\overline{PSW'}}(t; B, \omega, t', x) = \frac{B}{\omega}(t - t') + (1 - x)B$
Lower bound of first type PSW flow	$PSW(t; D, \omega, t', x) = D(t - t')$
Lower bound of second type PSW flow	$\underline{\underline{PSW'}}(t; B, \omega, t', x) = \frac{B}{\omega}(t - t')$

cial investment has a setup cost of A_n \$/transaction. This transaction cost is withdrawn from the cash storage when the financial investment is made, as is defined in CF6. This structure is motivated by Baumol's cash management model, which is in turn adapted from the economic order quantity (EOQ) model (Maness and Zietlow, 1998). In addition, we assume that the setup cost transactions of CF5, CF6 and CF7 and the inventory operating cost of CF8 are paid proportionally with material processing. In other words, the cash flows of the setup cost transactions and their material flows have the same cycle time, startup time and storage operation time fraction but different batch sizes. The cash flow of the inventory operating cost is proportional to the inventory level. Each cash flow is represented by the PSW model as follows

$$PSW(t;\, D,\, \omega,\, t',\, x) = D\omega\left[\operatorname{int}\left[\frac{t - t'}{\omega}\right] + \min\left\{1, \frac{1}{x}\, res\left[\frac{t - t'}{\omega}\right]\right\}\right] \quad (1)$$

or

$$PSW'(t;\, B,\, \omega,\, t',\, x) = B\left[\operatorname{int}\left[\frac{t - t'}{\omega}\right] + \min\left\{1, \frac{1}{x}\, res\left[\frac{t - t'}{\omega}\right]\right\}\right] \quad (2)$$

where D is the average flow rate, B is the batch size, ω is the cycle time, t' is the startup time, x is the storage operation time fraction, and t is time (Yi and Reklaitis 2002). Note that $D = B/\omega$. We refer to Eq. 1 as the first type of PSW flow and Eq. 2 as the second type of PSW flow. Note that average flow rate is used in the first type, whereas batch size is used in the second type. The two types of PSW flow have different upper/lower bounds and partial derivatives. Table 1 lists the expressions for the average and upper/lower bounds of the first and second types of PSW flow. Table 2 defines the variables and parameters of the PSW flows for CF4, CF9 and CF10. The average flow rate of sales tax is proportional to that of total sales revenue, that is $\hat{E} = \varsigma \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j$, where ς(\$/\$) is the sales tax rate. The cycle time, startup time and storage operation time fraction of sales tax are given parameters. Note that the sales tax startup time, $\hat{t}_m^j$, is the first taxing date after $t_m^j + \Delta t_m^j$.

Nonlinear Optimization Model

In order to obtain an analytical solution, we assume that stockholder dividends are paid in constant amounts but with variable frequency, that is

$$B_o \equiv \text{constant and } t_o = t_{o'} (\equiv t_{\bar{o}})\ \ \forall\, o \neq o' \in O \qquad (3)$$

Invoking this assumption simplifies the Lagrange multipliers in the solution using Kuhn-Tucker conditions. However, dividends to stockholders are typically paid yearly and at the same time, that is, $\omega_o \equiv$constant. Usually, the dividend cash flow is sufficiently small that the assumption in Eq. 3 is tolerable. If we choose a small value of B_o, the solution of this analysis gives a feasible near optimal solution. In order to find the exact solution of stockholder dividends, it is necessary to apply an iterative procedure in which B_o is varied until ω_o matches its true value. Note that, in this study, the total dividends to stockholders equal the net profit before tax.

All the other financial flows can be represented by previously defined variables and parameters. Table 3 lists the functional forms of CF1~CF10 obtained using Eqs. 1 and 2 with the defined variables and parameters. We define *C(0)* as the initial cash inventory and *C(t)* as the cash inventory at time t. Then, the cash inventory at time t is calculated by adding the incoming flows CF1 and CF2 to the initial cash inventory and subtracting the outgoing flows CF3~CF10

$$C(t) = C(0) + \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} (1 + \varsigma) S_m^j PSW(t;\, D_m^j,\, \omega_m^j,\, t_m^j + \Delta t_m^j,\, x_m^j) + \sum_{n=1}^{|N|} PSW(t;\, E_n(1 + \kappa_n \Delta t_n),\, \omega_n,\, t_n + \Delta t_n,\, x_n) - \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j PSW(t;\, D_k^j,\, \omega_k^j,\, t_k^j + \Delta t_k^j,\, x_k^j)$$

Table 2. Definition of PSW Parameters or Variables for Cash Flows

	CF4	CF9	CF10
Lot size		B_o	
Average flow rate	E_n		$\hat{E}$
Cycle time	ω_n	ω_o	$\hat{\omega}$
Startup time	t_n	t_o	$\hat{t}_m^j$
Storage operation time fraction	x_n	x_o	$\hat{x}$

Table 3. Mathematical Representation of Cash Flows

	PSW flows
CF1	$\sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}(1+\varsigma)S_m^j PSW(t;D_m^j,\omega_m^j,t_m^j+\Delta t_m^j,x_m^j)$
CF2	$\sum_{n=1}^{\lvert N\rvert} PSW(t;E_n(1+\kappa_n\Delta t_n),\omega_n,t_n+\Delta t_n,x_n)$
CF3	$\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}P_k^j PSW(t;D_k^j,\omega_k^j,t_k^j+\Delta t_k^j,x_k^j)$
CF4	$\sum_{n=1}^{\lvert N\rvert} PSW(t;E_n,\omega_n,t_n,x_n)$
CF5	$\sum_{j=1}^{\lvert J\rvert}\sum_{k\in\{D_k^j\}^+}^{\lvert K(j)\rvert} PSW'(t;A_k^j,\omega_k^j,t_k^j,x_k^j)$
CF6	$\sum_{n\in\{E_n\}^+}^{\lvert N\rvert} PSW'(t;A_n,\omega_n,t_n,x_n)$
CF7	$\sum_{i\in\{D_i\}^+}^{\lvert I\rvert} PSW'(t;A_i,\omega_i,t_i^{in},x_i^{in})$
CF8	$\sum_{j=1}^{\lvert J\rvert} h^j\int_0^t V^j(t)dt$
CF9	$\sum_{o=1}^{\lvert O\rvert} PSW'(t;B_o,\omega_o,t_o,x_o)$
CF10	$PSW\left(t;\varsigma\sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}S_m^jD_m^j,\hat{\omega},\hat{t}_m^j,\hat{x}\right)$

$$
\begin{aligned}
&-\sum_{n=1}^{|N|} PSW(t;E_n,\omega_n,t_n,x_n)\\
&-\sum_{j=1}^{|J|}\sum_{k\in\{D_k^j\}^+}^{|K(j)|} PSW'(t;A_k^j,\omega_k^j,t_k^j,x_k^j)\\
&-\sum_{n\in\{E_n\}^+}^{|N|} PSW'(t;A_n,\omega_n,t_n,x_n)\\
&-\sum_{i\in\{D_i\}^+}^{|I|} PSW'(t;A_i,\omega_i,t_i^{in},x_i^{in})-\sum_{j=1}^{|J|} h^j\int_0^t V^j(t)dt\\
&-\sum_{o=1}^{|O|} PSW'(t;B_o,\omega_o,t_o,x_o)\\
&-PSW\left(t;\varsigma\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|}S_m^jD_m^j,\hat{\omega},\hat{t}_m^j,\hat{x}\right) \qquad (4)
\end{aligned}
$$

where $\{D_k^j\}^+\equiv\{k|D_k^j>0\}$, $\{E_n\}^+\equiv\{n|E_n>0\}$ and $\{D_i\}^+\equiv\{i|D_i>0\}$, that is, the index sets with positive average flow rates. The average level of the cash inventory ($\bar{C}$), and the lower bound of the cash inventory ($\underline{\underline{C}}$) are easily calculated by using the equations in Table 1

$$
\begin{aligned}
\bar{C}=C(0)&+\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|}(1+\varsigma)S_m^j\overline{PSW}(t;D_m^j,\omega_m^j,t_m^j+\Delta t_m^j,x_m^j)\\
&-\sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}P_k^j\overline{PSW}(t;D_k^j,\omega_k^j,t_k^j+\Delta t_k^j,x_k^j)\\
&-\sum_{i\in\{D_i\}^+}^{|I|}\overline{PSW'}(t;A_i,\omega_i,t_i^{in},x_i^{in})\\
&-\sum_{j=1}^{|J|}\sum_{k\in\{D_k^j\}^+}^{|K(j)|}\overline{PSW'}(t;A_k^j,\omega_k^j,t_k^j,x_k^j)-\sum_{j=1}^{|J|}h^j\bar{V}^jt\\
&-\sum_{n=1}^{|N|}\overline{PSW}(t;E_n,\omega_n,t_n,x_n)\\
&+\sum_{n=1}^{|N|}\overline{PSW}(t;E_n(1+\kappa_n\Delta t_n),\omega_n,t_n+\Delta t_n,x_n)\\
&-\sum_{n\in\{E_n\}^+}^{|N|}\overline{PSW'}(t;A_n,\omega_n,t_n,x_n)\\
&-\sum_{o=1}^{|O|}\overline{PSW'}(t;B_o,\omega_o,t_o,x_o)\\
&-\overline{PSW}\left(t;\varsigma\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|}S_m^jD_m^j,\hat{\omega},\hat{t}_m^j,\hat{x}\right) \qquad (5)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\underline{C}}=C(0)&+\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|}(1+\varsigma)S_m^j\underline{\underline{PSW}}(t;D_m^j,\omega_m^j,t_m^j+\Delta t_m^j,x_m^j)\\
&-\sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}P_k^j\overline{\overline{PSW}}(t;D_k^j,\omega_k^j,t_k^j+\Delta t_k^j,x_k^j)\\
&-\sum_{i\in\{D_i\}^+}^{|I|}\overline{\overline{PSW'}}(t;A_i,\omega_i,t_i^{in},x_i^{in})\\
&-\sum_{j=1}^{|J|}\sum_{k\in\{D_k^j\}^+}^{|K(j)|}\overline{\overline{PSW'}}(t;A_k^j,\omega_k^j,t_k^j,x_k^j)-\sum_{j=1}^{|J|}h^j\overline{\overline{V}}^jt\\
&-\sum_{n=1}^{|N|}\overline{\overline{PSW}}(t;E_n,\omega_n,t_n,x_n)+\sum_{n=1}^{|N|}\underline{\underline{PSW}}\\
&\times(t;E_n(1+\kappa_n\Delta t_n),\omega_n,t_n+\Delta t_n,x_n)\\
&-\sum_{n\in\{E_n\}^+}^{|N|}\overline{\overline{PSW'}}(t;A_n,\omega_n,t_n,x_n)
\end{aligned}
$$

$$- \sum_{o=1}^{|O|} \overline{\overline{PSW'}}(t; B_o, \omega_o, t_o, x_o)$$

$$- \overline{\overline{PSW}}\left(t; \varsigma \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j, \hat{\omega}, \hat{t}_m^j, \hat{x}\right) \quad (6)$$

We assume that the cash in-flows and out-flows are balanced in the long run. The average flow rates of cash flows into and out of the cash storage unit satisfy the following balance equation

$$\sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j + \sum_{n=1}^{|N|} \kappa_n \Delta t_n E_n = \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j D_k^j + \sum_{i \in \{D_i\}^+}^{|I|} \frac{A_i}{\omega_i} + \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} \frac{A_k^j}{\omega_k^j} + \sum_{n \in \{E_n\}^+}^{|N|} \frac{A_n}{\omega_n} + \sum_{j=1}^{|J|} h^j \overline{V^j} + \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \quad (7)$$

Then, by using the equations in Table 1 and Eq. 7, Eqs. 5 and 6 can be simplified to the following forms

$$\bar{C} = C(0) + \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} (1 + \varsigma) S_m^j [0.5(1 - x_m^j) D_m^j \omega_m^j - D_m^j (t_m^j + \Delta t_m^j)] - \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j [0.5(1 - x_k^j) D_k^j \omega_k^j - D_k^j (t_k^j + \Delta t_k^j)] - \sum_{i \in \{D_i\}^+}^{|I|} A_i \left[0.5(1 - x_i^{in}) - \frac{t_i^{in}}{\omega_i}\right] - \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} A_k^j \left[0.5(1 - x_k^j) - \frac{t_k^j}{\omega_k^j}\right] - \sum_{n \in \{E_n\}^+}^{|N|} A_n \left[0.5(1 - x_n) - \frac{t_n}{\omega_n}\right] - \sum_{n=1}^{|N|} [\kappa_n \Delta t_n t_n + \kappa_n (\Delta t_n)^2 + \Delta t_n - 0.5(1 - x_n) \kappa_n \Delta t_n \omega_n] E_n + \sum_{o=1}^{|O|} \left[\frac{t_{\bar{o}} B_o}{\omega_o} - 0.5 B_o (1 - x_o)\right] - \varsigma \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j [0.5(1 - \hat{x}) \hat{\omega} - \hat{t}_m^j] \quad (8)$$

$$\underline{\underline{C}} = C(0) - \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} (1 + \varsigma) S_m^j [D_m^j (t_m^j + \Delta t_m^j)] - \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j [(1 - x_k^j) D_k^j \omega_k^j - D_k^j (t_k^j + \Delta t_k^j)] - \sum_{i \in \{D_i\}^+}^{|I|} A_i \left[(1 - x_i^{in}) - \frac{t_i^{in}}{\omega_i}\right] - \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} A_k^j \left[(1 - x_k^j) - \frac{t_k^j}{\omega_k^j}\right] - \sum_{j=1}^{|J|} h^j [\overline{\overline{V^j}} - \overline{V^j}] - \sum_{n=1}^{|N|} [\kappa_n \Delta t_n t_n + \kappa_n (\Delta t_n)^2 + \Delta t_n + (1 - x_n) \omega_n] E_n - \sum_{n \in \{E_n\}^+}^{|N|} A_n \left[(1 - x_n) - \frac{t_n}{\omega_n}\right] + \sum_{o=1}^{|O|} \left[\frac{t_{\bar{o}} B_o}{\omega_o} - B_o (1 - x_o)\right] - \varsigma \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j [(1 - \hat{x}) \hat{\omega} - \hat{t}_m^j] \quad (9)$$

Note that the term $\sum_{o=1}^{|O|} B_o/\omega_o$ in Eqs. 8 and 9 can be further developed by Eq. 7. Equation 8 will be used to compute the opportunity cost of the cash inventory. Equation 9 should be greater than or equal to zero in order for the cash storage not to be depleted, because a shortage of cash will incur severe additional costs or bankruptcy. Therefore, $V^j \geq 0$ and $\underline{\underline{C}} \geq 0$ constitute the constraints of the design optimization.

Suppose η (\$/\$/year) is the rate of opportunity cost of the cash inventory. The objective function of the optimization is to minimize the annualized opportunity costs of capital investment for process/storage units and cash/material inventories minus the dividend to stockholders:

$$Minimize\ TC = \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} [a_k^j D_k^j \omega_k^j] + \sum_{i=1}^{|I|} [a_i D_i \omega_i] + \sum_{n=1}^{|N|} [a_n E_n \omega_n] + \sum_{j=1}^{|J|} [b^j \overline{\overline{V^j}}] + \eta \bar{C} + \sum_{j=1}^{|J|} \gamma^j \overline{V^j} - \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \quad (10)$$

where a_n (\$/\$/year) is the annualized financial investment cost, which is proportional to size of the temporary financial investment. Variable a_N is introduced for the mathematical analogy to a_k^j and a_i. By using Eq. 7, Eq. 10 can be rewritten as

$$TC = \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \left[\frac{A_k^j}{\omega_k^j} + a_k^j D_k^j \omega_k^j + P_k^j D_k^j\right] + \sum_{i=1}^{|I|} \left[\frac{A_i}{\omega_i} + a_i D_i \omega_i\right] + \sum_{n=1}^{|N|} \left[\frac{A_n}{\omega_n} + a_n E_n \omega_n\right] + \sum_{j=1}^{|J|} [(h^j + \gamma^j) \overline{V^j} + b^j \overline{\overline{V^j}}] - \sum_{n=1}^{|N|} \kappa_n \Delta t_n E_n - \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j + \eta \bar{C} \quad (11)$$

Without loss of generality, the storage size will be determined by the upper bound of the inventory holdup $\overline{\overline{V^j}}$. Therefore, Eq. A4 is the expression for the storage capacity. The independent variables are selected to be the cycle times (ω_k^j, ω_i and ω_n), start-up times (t_k^j, t_i^{in} and t_n), and average material/cash flow rates (D_k^j, D_i and E_n). Note that the start-up time t_i^{out} is converted into t_i^{in} by Eq. A1.

The objective function Eq. 11 is convex, and the constraints are linear with respect to ω_k^j, ω_i, ω_n, t_k^j, t_i^{in} and t_n if D_k^j, D_i and E_n are considered as parameters. However, the convexity with respect to D_k^j, D_i and E_n is not clear. First, we obtain the

Table 4. Optimization Problem and Solution of Kuhn-Tucker Conditions

Objective Function	$TC = \sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\left[\frac{A_k^j}{\omega_k^j} + a_k^j D_k^j \omega_k^j + P_k^j D_k^j\right] + \sum_{i=1}^{\lvert I\rvert}\left[\frac{A_i}{\omega_i} + a_i D_i \omega_i\right] + \sum_{n=1}^{\lvert N\rvert}\left[\frac{A_n}{\omega_n} + a_n E_n \omega_n\right] + \sum_{j=1}^{\lvert J\rvert}[(h^j + \gamma^j)\overline{V^j}$ $+ b^j\overline{\overline{V^j}}] - \sum_{n=1}^{\lvert N\rvert}\kappa_n \Delta t_n E_n - \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert} S_m^j D_m^j + \eta\bar{C}$
Design Variables	$\omega_k^t,\ \omega_i,\ \omega_n,\ t_k^j,\ t_i^{in},\ t_n,\ D_k^j,\ D_i,\ E_n$
Constraints	$\sum_{i=1}^{\lvert I\rvert} g_i^j D_i + \sum_{k=1}^{\lvert K(j)\rvert} D_k^j = \sum_{i=1}^{\lvert I\rvert} f_i^j D_i + \sum_{m=1}^{\lvert M(j)\rvert} D_m^j,\ \underline{\underline{V^j}} \geq 0$ and $\underline{\underline{C}} \geq 0$
Solution of Kuhn-Tucker Conditions with Fixed D_k^j, D_i, E_n	$\omega_k^j = \sqrt{\frac{A_k^j}{D_k^j \Psi_k^j}},\quad \omega_i = \sqrt{\frac{A_i}{D_i \Psi_i}},\quad \omega_n = \sqrt{\frac{A_n}{E_n \Psi_n}}$ $\sum_{k=1}^{\lvert K(j)\rvert} D_k^j t_k^j + \sum_{i=1}^{\lvert I\rvert}(g_i^j - f_i^j) D_i t_i^{in} = V^j(0) - \sum_{m=1}^{\lvert M(j)\rvert}(1 - x_m^j) D_m^j \omega_m^j + \sum_{m=1}^{\lvert M(j)\rvert} D_m^j t_m^j - \sum_{i=1}^{\lvert I\rvert}[(1 - x_i^{in}) f_i^j + (1 - x_i^{out}) g_i^j] D_i \omega_i$ $^*TC(D_k^j, D_i, E_n) = 2\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\sqrt{A_k^j \Psi_k^j D_k^j} + \sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert} P_k^j D_k^j + 2\sum_{i=1}^{\lvert I\rvert}\sqrt{A_i \Psi_i D_i} + 2\sum_{n=1}^{\lvert N\rvert}\sqrt{A_n \Psi_n E_n}$ $- \sum_{n=1}^{\lvert N\rvert}\kappa_n \Delta t_n E_n + \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}\left[\left(\frac{h^j + \gamma^j + \eta h^j + \eta(1 + \varsigma)S_m^j}{2} + b^j\right)(1 - x_m^j)\omega_m^j - S_m^j\right] D_m^j + \eta\sum_{i\in\{D_i\}^+}^{\lvert I\rvert}$ $\times 0.5(1 - x_i^{in}) A_i + \eta\sum_{j=1}^{\lvert J\rvert}\sum_{k\in\{D_k^j\}^+}^{\lvert K(j)\rvert} 0.5(1 - x_k^j) A_k^j + \eta\sum_{n\in\{E_n\}^+}^{\lvert N\rvert} 0.5(1 - x_n) A_n + \eta\sum_{o=1}^{\lvert O\rvert} 0.5 B_o (1 - x_o)$ $+ \eta\sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert} 0.5 S_m^j D_m^j \varsigma(1 - \hat{x})\hat{\omega}$ $\sum_{n=1}^{\lvert N\rvert}\left\{(t_{\bar{o}} - t_n) + \frac{(1 - x_n)}{\Psi_n}\right\}\sqrt{A_n \Psi_n E_n} + \sum_{n=1}^{\lvert N\rvert}[\Delta t_n(1 + \kappa_n \Delta t_n) - \kappa_n \Delta t_n (t_{\bar{o}} - t_n)] E_n = C(0)$ $+ \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert} S_m^j D_m^j [t_{\bar{o}} - (1 + \varsigma)(t_m^j + \Delta t_m^j) - \varsigma\{(1 - \hat{x})\hat{\omega} - \hat{t}_m^j\}] - \sum_{i\in\{D_i\}^+}^{\lvert I\rvert}[A_i(1 - x_i^{in})$ $+ \sqrt{A_i \Psi_i D_i}(t_{\bar{o}} - t_i^{in})] - \sum_{j=1}^{\lvert J\rvert}\sum_{k\in\{D_k^j\}^+}^{\lvert K(j)\rvert}[P_k^j D_k^j\{(1 - x_k^j)\omega_k^j + (t_{\bar{o}} - t_k^j - \Delta t_k^j)\} + A_k^j(1 - x_k^j)$ $+ \sqrt{A_k^j \Psi_k^j D_k^j}(t_{\bar{o}} - t_k^j)] - \sum_{j=1}^{\lvert J\rvert}(1 + t_{\bar{o}}) h^j {}^*\overline{V^j} - \sum_{o=1}^{\lvert O\rvert} B_o(1 - x_o) - \sum_{n\in\{E_n\}^+}^{\lvert N\rvert} A_n(1 - x_n)$

solution for Kuhn-Tucker conditions with respect to ω_k^j, ω_i, ω_n, t_k^j, t_i^{in} and t_n when D_k^j, D_i and E_n are considered as parameters, and then, we further solve the problem with respect to D_k^j, D_i and E_n. Although the problem is separated into a two-level parametric optimization problem, the Kuhn-Tucker conditions of the original problem, and the two-level problem are the same if the constraints are reduced to equality (Appendix B). In other words, the Kuhn-Tucker conditions of the first problem produce an explicit analytical solution and the original problem can be reduced to the second problem by eliminating the design variables of the first problem. The first level of the two-level problem has a convex objective with linear inequality constraints and the second level has nonconvex objective with nonlinear equality constraints. The two-level parametric approach yields a global optimum inasmuch as the second problem converges to its global optimum point. Table 4 shows a summary of the optimization problem along with the solution of the Kuhn-Tucker conditions of the first problem.

Solution of Kuhn-Tucker Conditions

The solution of the Kuhn-Tucker conditions of the first level optimization problem, which entails minimizing the objective function Eq. 11 subject to the constraints $\underline{\underline{V^j}} \geq 0$ and $\underline{\underline{C}} \geq 0$ and with fixed values of D_k^j, D_i and E_n, is obtained by the algebraic manipulation summarized in Appendix C (see citation to Table 5 in Appendix C). Optimal cycle times are

$$\omega_k^j = \sqrt{\frac{A_k^j}{D_k^j \Psi_k^j}} \tag{12}$$

$$\omega_i = \sqrt{\frac{A_i}{D_i \Psi_i}} \tag{13}$$

$$\omega_n = \sqrt{\frac{A_n}{E_n \Psi_n}} \tag{14}$$

where

$$\Psi_k^j = \left(\frac{h^j + \gamma^j + \eta h^j + \eta P_k^j}{2} + b^j\right)(1 - x_k^j) + a_k^j \quad (15)$$

$$\Psi_i = a_i + (1 - x_i^{in}) \sum_{j=1}^{|J|} \left(\frac{h^j + \gamma^j + \eta h^j}{2} + b^j\right) f_i^j + (1 - x_i^{out}) \sum_{j=1}^{|J|} \left(\frac{h^j + \gamma^j + \eta h^j}{2} + b^j\right) g_i^j \quad (16)$$

$$\Psi_n = a_n + \eta(1 - x_n)(1 + 0.5\kappa_n \Delta t_n) \quad (17)$$

Note that, due to the financial cost factors, the optimal lot sizes of Eqs. 12 and 13 are smaller than those derived previously using approaches such as the classical EOQ model. Because the values of the multipliers are positive, Eq. C8 gives $\underline{\underline{V^j}} = 0$ and $\underline{\underline{C}} = 0$. From Eqs. A5 and 9, we obtain the following expressions

$$\sum_{k=1}^{|K(j)|} D_k^j t_k^j + \sum_{i=1}^{|I|} (g_i^j - f_i^j) D_i t_i^{in} = V^j(0) - \sum_{m=1}^{|M(j)|} (1 - x_m^j) D_m^j \omega_m^j + \sum_{m=1}^{|M(j)|} D_m^j t_m^j - \sum_{i=1}^{|I|} [(1 - x_i^{in}) f_i^j + (1 - x_i^{out}) g_i^j] D_i \omega_i \quad (18)$$

$$\sum_{n=1}^{|N|} \left\{(t_{\bar{o}} - t_n) + \frac{(1 - x_n)}{\Psi_n}\right\} \sqrt{A_n \Psi_n E_n} + \sum_{n=1}^{|N|} [\Delta t_n (1 + \kappa_n \Delta t_n) - \kappa_n \Delta t_n (t_{\bar{o}} - t_n)] E_n = C(0) + \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j [t_{\bar{o}} - (1 + \varsigma) \times (t_m^j + \Delta t_m^j) - \varsigma\{(1 - \hat{x})\hat{\omega} - \hat{t}_m^j\}] - \sum_{i \in \{D_i\}^+}^{|I|} [A_i (1 - x_i^{in}) + \sqrt{A_i \Psi_i D_i} (t_{\bar{o}} - t_i^{in})] - \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} [P_k^j D_k^j \{(1 - x_k^j)\omega_k^j + (t_{\bar{o}} - t_k^j - \Delta t_k^j)\} + A_k^j (1 - x_k^j) + \sqrt{A_k^j \Psi_k^j D_k^j} (t_{\bar{o}} - t_k^j)] - \sum_{j=1}^{|J|} (1 + t_{\bar{o}}) h^j (*\overline{V^j}) - \sum_{o=1}^{|O|} B_o (1 - x_o) - \sum_{n \in \{E_n\}^+}^{|N|} A_n (1 - x_n) \quad (19)$$

where

$$*\overline{V^j} = \sum_{k=1}^{|K(j)|} \frac{(1 - x_k^j)}{2} D_k^j \omega_k^j + \sum_{m=1}^{|M(j)|} \frac{(1 - x_m^j)}{2} D_m^j \omega_m^j + \sum_{i=1}^{|I|} \frac{(1 - x_i^{out})}{2} g_i^j D_i \omega_i + \sum_{i=1}^{|I|} \frac{(1 - x_i^{in})}{2} f_i^j D_i \omega_i \quad (20)$$

Equation 20 is derived from Eqs. A6 and 18. From Eqs. A4 and 18, we find that the optimal material storage size is $*\overline{\overline{V^j}} = 2*\overline{V^j}$. The optimal average level of cash storage, calculated using Eqs. 8 and 19, is as follows

$$*\bar{C} = \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} 0.5 S_m^j D_m^j [(1 + \varsigma)(1 - x_m^j)\omega_m^j + \varsigma(1 - \hat{x})\hat{\omega}] + \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j [0.5(1 - x_k^j) D_k^j \omega_k^j] + \sum_{i \in \{D_i\}^+}^{|I|} 0.5 A_i (1 - x_i^{in}) + \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} 0.5 A_k^j (1 - x_k^j) + \sum_{n \in \{E_n\}^+}^{|N|} 0.5 A_n (1 - x_n) + \sum_{j=1}^{|J|} h^j * \overline{V^j} + \sum_{n=1}^{|N|} (1 - x_n)(1 + 0.5\kappa_n \Delta t_n)\omega_n E_n + \sum_{o=1}^{|O|} 0.5 B_o (1 - x_o) \quad (21)$$

Thus, optimal cash storage size is $*\bar{\bar{C}} = 2*\bar{C}$. Then, the optimal objective value, calculated from Eqs. 11, 12 ~17 and 21, is as follows

$$*TC(D_k^j, D_i, E_n) = 2 \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} \sqrt{A_k^j \Psi_k^j D_k^j} + \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} P_k^j D_k^j + 2 \sum_{i=1}^{|I|} \sqrt{A_i \Psi_i D_i} + 2 \sum_{n=1}^{|N|} \sqrt{A_n \Psi_n E_n} - \sum_{n=1}^{|N|} \kappa_n \Delta t_n E_n + \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} \left[\left(\frac{h^j + \gamma^j + \eta h^j + \eta(1 + \varsigma) S_m^j}{2} + b^j\right)(1 - x_m^j)\omega_m^j - S_m^j\right] D_m^j + \eta \sum_{i \in \{D_i\}^+}^{|I|} 0.5(1 - x_i^{in}) A_i + \eta \sum_{j=1}^{|J|} \sum_{k \in \{D_k^j\}^+}^{|K(j)|} 0.5 (1 - x_k^j) A_k^j + \eta \sum_{n \in \{E_n\}^+}^{|N|} 0.5(1 - x_n) A_n + \eta \sum_{o=1}^{|O|} 0.5 B_o (1 - x_o) + \eta \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} 0.5 S_m^j D_m^j \varsigma (1 - \hat{x})\hat{\omega} \quad (22)$$

By using Eqs. 12 ~ 17, Eq. 7 can be rewritten as

$$\sum_{o=1}^{|O|} \frac{B_o}{\omega_o} = \sum_{j=1}^{|J|} \sum_{m=1}^{|M(j)|} S_m^j D_m^j - \sum_{j=1}^{|J|} \sum_{k=1}^{|K(j)|} [\sqrt{A_k^j \Psi_k^j D_k^j} + P_k^j D_k^j] - \sum_{i=1}^{|I|} \sqrt{A_i \Psi_i D_i} - \sum_{n=1}^{|N|} \sqrt{A_n \Psi_n E_n} + \sum_{n=1}^{|N|} \kappa_n \Delta t_n E_n - \sum_{j=1}^{|J|} h^j (*\overline{V^j}) \quad (23)$$

Table 5. Partial Derivatives of Important Functions Involved in Langrangian

$X \equiv$	t_k^k	t_i^{in}	t_n
$\frac{\partial \overline{\overline{V}}^j}{\partial X} =$	$-D_k^j$	$(f_i^j - g_i^j)D_i$	
$\frac{\partial \overline{V}^j}{\partial X} =$	$-D_k^j$	$(f_i^j - g_i^j)D_i$	
$\frac{\partial \underline{\underline{V}}^j}{\partial X} =$	$-D_k^j$	$(f_i^j - g_i^j)D_i$	
$\frac{\partial}{\partial X}\left(\bar{C} - t_{\bar{o}} \sum_{o=1}^{\lvert O \rvert} \frac{B_o}{\omega_o}\right) =$	$P_k^j D_k^j + \frac{A_k^j}{\omega_k^j}$	$\frac{A_i}{\omega_i}$	$-\kappa_n \Delta t_n E_n + \frac{A_n}{\omega_n}$
$\frac{\partial}{\partial X}\left(\underline{\underline{C}} - t_{\bar{o}} \sum_{o=1}^{\lvert O \rvert} \frac{B_o}{\omega_o}\right) =$	$P_k^j D_k^j + \frac{A_k^j}{\omega_k^j}$	$\frac{A_i}{\omega_i}$	$-\kappa_n \Delta t_n E_n + \frac{A_n}{\omega_n}$

$X \equiv$	ω_k^j	ω_j	ω_n
$\frac{\partial \overline{\overline{V}}^j}{\partial X} =$	$(1 - x_k^j)D_k^j$		
$\frac{\partial \overline{V}^j}{\partial X} =$	$\frac{(1 - x_k^j)}{2} D_k^j$	$-\frac{(1 - x_i^{in})}{2} f_i^j D_i - \frac{(1 - x_i^{out})}{2} g_i^j D_i$	
$\frac{\partial \underline{\underline{V}}^j}{\partial X} =$		$-(1 - x_i^{in}) f_i^j D_i - (1 - x_i^{out}) g_i^j D_i$	
$\frac{\partial}{\partial X}\left(\bar{C} - t_{\bar{o}} \sum_{o=1}^{\lvert O \rvert} \frac{B_o}{\omega_o}\right) =$	$-\frac{(P_k^j)(1 - x_k^j)}{2} D_k^j - \frac{A_k^j t_k^j}{(\omega_k^j)^2}$	$-\frac{A_i t_i^{in}}{(\omega_i)^2}$	$0.5(1 - x_n)E_n \kappa_n \Delta t_n - \frac{A_n t_n}{(\omega_n)^2}$
$\frac{\partial}{\partial X}\left(\underline{\underline{C}} - t_{\bar{o}} \sum_{o=1}^{\lvert O \rvert} \frac{B_o}{\omega_o}\right) =$	$-(0.5h^j + P_k^j)(1 - x_k^j)D_k^j - \frac{A_k^j t_k^j}{(\omega_k^j)^2}$	$-\sum_{j=1}^{\lvert J \rvert} h^j \left[\frac{(1 - x_i^{in})}{2} f_i^j D_i + \frac{(1 - x_i^{out})}{2} g_i^j D_i\right] - \frac{A_i t_i^{in}}{(\omega_i)^2}$	$-(1 - x_n)E_n - \frac{A_n t_n}{(\omega_n)^2}$

The second level optimization problem, which entails minimizing the objective function of Eq. 22 under the constraints of Eqs. A2, 18, 19 and 20 with respect to the design variables, D_k^j, D_i, E_n and t_n, is summarized in Table 6. The second level optimization problem is a nonconvex nonlinear programming with bilinear terms, $D_k^j t_k^j$ and $D_i t_i^{in}$, as well as separable con-

Table 6. Second Level Optimization Problem

Objective function

$$*TC(D_k^j, D_t, E_n) = 2\sum_{j=1}^{\lvert J \rvert}\sum_{k=1}^{\lvert K(j) \rvert} \sqrt{A_k^j \Psi_k^j D_k^j} + \sum_{j=1}^{\lvert J \rvert}\sum_{k=1}^{\lvert K(j) \rvert} P_k^j D_k^j + 2\sum_{i=1}^{\lvert I \rvert} \sqrt{A_i \Psi_i D_i} + 2\sum_{n=1}^{\lvert N \rvert} \sqrt{A_n \Psi_n E_n} - \sum_{n=1}^{\lvert N \rvert} \kappa_n \Delta t_n E_n$$

$$+ \sum_{j=1}^{\lvert J \rvert}\sum_{m=1}^{\lvert M(j) \rvert} \left[\left(\frac{h^j + \gamma^j + \eta h^j + \eta(1 + \varsigma)S_m^j}{2} + b^j\right)(1 - x_m^j)\omega_m^j - S_m^j\right] D_m^j + \eta \sum_{i \in \{D_i\}^+}^{\lvert I \rvert} 0.5(1 - x_i^{in})A_i$$

$$+ \eta \sum_{j=1}^{\lvert J \rvert}\sum_{k \in \{D_k^j\}^+}^{\lvert K(j) \rvert} 0.5(1 - x_k^j)A_k^j + \eta \sum_{n \in \{E_n\}^+}^{\lvert N \rvert} 0.5(1 - x_n)A_n + \eta \sum_{o=1}^{\lvert O \rvert} 0.5B_o(1 - x_o) + \eta \sum_{j=1}^{\lvert J \rvert}\sum_{m=1}^{\lvert M(j) \rvert} 0.5 S_m^j D_m^j \varsigma(1 - \hat{x})\hat{\omega}$$

Design Variables

D_k^j, D_t, E_n and t_n

Constraints

$$\sum_{i=1}^{\lvert I \rvert} g_i^j D_i + \sum_{k=1}^{\lvert K(j) \rvert} D_k^j = \sum_{i=1}^{\lvert I \rvert} f_i^j D_i + \sum_{m=1}^{\lvert M(j) \rvert} D_m^j$$

$$\sum_{n=1}^{\lvert N \rvert} \left\{(t_{\bar{o}} - t_n) + \frac{(1 - x_n)}{\Psi_n}\right\} \sqrt{A_n \Psi_n E_n} + \sum_{n=1}^{\lvert N \rvert} [\Delta t_n(1 + \kappa_n \Delta t_n) - \kappa_n \Delta t_n(t_{\bar{o}} - t_n)]E_n = C(0) + \sum_{j=1}^{\lvert J \rvert}\sum_{m=1}^{\lvert M(j) \rvert} S_m^j D_m^j [t_{\bar{o}}$$

$$- (1 + \varsigma)(t_m^j + \Delta t_m^j) - \varsigma\{(1 - \hat{x})\hat{\omega} - \hat{t}_m^j\}] - \sum_{i \in \{D_i\}^+}^{\lvert I \rvert} [A_i(1 - x_i^{in}) + \sqrt{A_i \Psi_i D_i}(t_{\bar{o}} - t_i^{in})] - \sum_{j=1}^{\lvert J \rvert}\sum_{k \in \{D_k^j\}^+}^{\lvert K(j) \rvert} [P_k^j D_k^j \{(1 - x_k^j)\omega_k^j$$

$$+ (t_{\bar{o}} - t_k^j - \Delta t_k^j)\} + A_k^j(1 - x_k^j) + \sqrt{A_k^j \Psi_k^j D_k^j}(t_{\bar{o}} - t_k^j)] - \sum_{j=1}^{\lvert J \rvert} (1 + t_{\bar{o}}) h^j * \overline{V^j} - \sum_{o=1}^{\lvert O \rvert} B_o(1 - x_o) - \sum_{n \in \{E_n\}^+}^{\lvert N \rvert} A_n(1 - x_n)$$

$$\sum_{k=1}^{\lvert K(j) \rvert} D_k^j t_k^j + \sum_{i=1}^{\lvert I \rvert} (g_i^j - f_i^j) D_i t_i^{in} = V^j(0) - \sum_{m=1}^{\lvert M(j) \rvert} (1 - x_m^j) D_m^j \omega_m^j + \sum_{m=1}^{\lvert M(j) \rvert} D_m^j t_m^j - \sum_{i=1}^{\lvert I \rvert} [(1 - x_i^{in}) f_i^j + (1 - x_i^{out}) g_i^j] \sqrt{\frac{A_i D_i}{\Psi_i}}$$

cave terms (square roots). Because some of the average flow rates will be zero at the optimum, it is not easy to compute the derivatives of their square roots at the optimum. Moreover, A_i, A_k^j and A_n should be zero if their corresponding average flow rates go to zero at the optimum. To address this issue, the objective function Eq. 22 should include binary variables to exclude the setup costs whose average flow rates become zero. In the following section, we introduce a suboptimal approach to reduce the computational complexity. The second level optimization problem can be replaced with another model (for example, ordinary linear programming) to compute the average rates of material and cash flows without damaging the optimality of the lot sizing equations derived from the first level optimization problem.

Additional Constraints

Note that the value of $2\sqrt{A_n \Psi_n E_n} - \kappa_n \Delta t_n E_n$ in Eq. 22 should be nonpositive; otherwise will be zero at the optimum point

$$\frac{4A_n\Psi_n}{(\kappa_n \Delta t_n)^2} \le E_n \tag{24}$$

The condition $\underline{\underline{C}} \ge 0$ should be satisfied even in the absence of temporary financial investments. For $|N| = 0$, Eq. 9 gives the following inequality with respect to $t_{\bar{o}}$

$$t_{\bar{o}} \ge \frac{\begin{array}{c} -C(0) + \sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|}(1+\varsigma)S_m^j D_m^j(t_m^j + \Delta t_m^j) + \sum_{i\in\{D_i\}^+}^{|I|}[A_i(1-x_i^{in}) - \sqrt{A_i\Psi_i D_i}(t_i^{in})] + \sum_{j=1}^{|J|}\sum_{k\in\{D_k^j\}^+}^{|K(j)|}[P_k^j D_k^j\{(1-x_k^j)\omega_k^j \\ - (t_k^j + \Delta t_k^j) + A_k^j(1-x_k^j) - \sqrt{A_k^j\Psi_k^j D_k^j}(t_k^j) + \sum_{j=1}^{|J|} h^j * \overline{V^j} + \sum_{o=1}^{|O|} B_o(1-x_o) + \varsigma\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|} S_m^j D_m^j[(1-\hat{x})\hat{\omega} - \hat{t}_m^j] \end{array}}{\sum_{j=1}^{|J|}\sum_{m=1}^{|M(j)|} S_m^j D_m^j - \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}[\sqrt{A_k^j\Psi_k^j D_k^j} + P_k^j D_k^j] - \sum_{i=1}^{|I|}\sqrt{A_i\Psi_i D_i} - \sum_{j=1}^{|J|} h^j * \overline{V^j}} \tag{25}$$

The equality in Eq. 25 gives the optimal value of $t_{\bar{o}}$ for a given $C(0)$ and $|N| = 0$. In practice, this value is very favorable because stockholders usually expect to be paid as soon as possible. When $t_{\bar{o}}$ is strictly greater than the right-hand side of Eq. 25, excess cash builds up. This excess cash can be utilized for temporary financial investments. Note that $t_{\bar{o}}$ is a design variable like t_n; however, for simplicity, this article handles it as a parameter. Unfortunately, neither Eq. 19 nor 25, guarantees the initial shortage of cash inventory mostly caused from the payment of raw material purchase before the income of product sales. The initial cash inventory *C(0)* should cover such a cash shortage. Let us define $t_c \equiv \min\{t_m^j + \Delta t_m^j\}$. Then

$$C(0) \ge \sum_{j=1}^{|J|}\sum_{k\in\{D_k^j\}^+}^{|K(j)|}[P_k^j PSW(t_c; D_k^j, \omega_k^j, t_k^j + \Delta t_k^j, x_k^j) + PSW'(t_c; A_k^j, \omega_k^j, t_k^j, x_k^j)] + \sum_{i\in\{D_i\}^+}^{|I|} PSW'(t_c; A_i, \omega_i, t_i^{in}, x_i^{in}) + \sum_{j=1}^{|J|} h^j \int_0^{t_c} V^j(t)dt \tag{26}$$

Minimizing the initial cash inventory is one of the hidden design criteria. *C(0)* can be determined from the equality in Equation 26, which may include additional terms from Eq. 4 depending on the circumstances. If *C(0)* is a given parameter and has an insufficient value to cover the initial cash requirement, Eq. 26 acts as a severe constraint with respect to the design variables. In such circumstances, the optimization problem will have a different solution. In order for the form of Eq. 26 to be suitable for use as a constraint of the optimization problem, the right-hand side of the equation should be further developed by using the expression for the upper bound of PSW flow shown in Table 1.

If the cycle times and/or startup times are outside the upper and/or lower bounds, either the bound closest to the unbounded optimal solution should be selected because the objective function is convex with respect to the cycle times and start-up times in the first optimization problem. In addition, the upper and lower bounds of the average flow rates can be easily incorporated into the separable programming that will be introduced in the following section.

A Suboptimal Approach

To reduce the computational burden, the second level problem can be decomposed into two subproblems as a suboptimal approach. The nonconstant terms in Eq. 22 can be separated into the terms with respect to D_k^j, D_i and the terms with respect to E_n.

$$TC1(D_k^j, D_i) = 2\sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|}\sqrt{A_k^j\Psi_k^j D_k^j} + \sum_{j=1}^{|J|}\sum_{k=1}^{|K(j)|} P_k^j D_k^j + 2\sum_{i=1}^{|I|}\sqrt{A_i\Psi_i D_i} \tag{27}$$

Table 7. Suboptimal Design Procedure Resulted from Kuhn-Tucker Conditions

	Variables	Subproblems
1	$D_k^j,\ D_i$	Minimize $*TC1(D_k^j, D_i) = 2\sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}\sqrt{A_k^j\Psi_k^jD_k^j} + \sum_{j=1}^{\lvert J\rvert}\sum_{k=1}^{\lvert K(j)\rvert}P_k^jD_k^j + 2\sum_{i=1}^{\lvert I\rvert}\sqrt{A_i\Psi_iD_i}$ such that; $\sum_{i=1}^{\lvert I\rvert}g_i^jD_i + \sum_{k=1}^{\lvert K(j)\rvert}D_k^j = \sum_{i=1}^{\lvert I\rvert}f_i^jD_i + \sum_{m=1}^{\lvert M(j)\rvert}D_m^j$
2	$\omega_k^j,\ \omega_i$	$\omega_k^j = \sqrt{\frac{A_k^j}{D_k^j\Psi_k^j}},\ \omega_i = \sqrt{\frac{A_i}{D_i\Psi_i}}$
	$t_k^j,\ t_i^{in}$	$\sum_{k=1}^{\lvert K(j)\rvert}D_k^jt_k^j + \sum_{i=1}^{\lvert I\rvert}(g_i^j - f_i^j)D_it_i^{in} = V^j(0) - \sum_{m=1}^{\lvert M(j)\rvert}(1 - x_m^j)D_m^j\omega_m^j + \sum_{m=1}^{\lvert M(j)\rvert}D_m^jt_m^j - \sum_{i=1}^{\lvert I\rvert}[(1 - x_i^{in})f_i^j + (1 - x_i^{out})g_i^j]D_i\omega_i$
3	$E_n,\ t_n$	Minimize $*TC2(E_n) = 2\sum_{n=1}^{\lvert N\rvert}\sqrt{A_n\Psi_nE_n} - \sum_{n=1}^{\lvert N\rvert}\kappa_n\Delta t_nE_n$ such that; $\sum_{n=1}^{\lvert N\rvert}\left\{(t_{\bar{o}} - t_n) + \frac{(1 - x_n)}{\Psi_n}\right\}\sqrt{A_n\Psi_nE_n} + \sum_{n=1}^{\lvert N\rvert}[\Delta t_n(1 + \kappa_n\Delta t_n) - \kappa_n\Delta t_n(t_{\bar{o}} - t_n)]E_n = C(0) + \sum_{j=1}^{\lvert J\rvert}\sum_{m=1}^{\lvert M(j)\rvert}S_m^jD_m^j[t_{\bar{o}} - (1 + \varsigma)(t_m^j + \Delta t_m^j) - \varsigma\{(1 - \hat{x})\hat{\omega} - \hat{t}_m^j\}] - \sum_{i\in\{D_i\}^+}^{\lvert I\rvert}[A_i(1 - x_i^{in}) + \sqrt{A_i\Psi_iD_i}(t_{\bar{o}} - t_i^{in})] - \sum_{j=1}^{\lvert J\rvert}\sum_{k\in\{D_k^j\}^+}^{\lvert K(j)\rvert}[P_k^jD_k^j\{(1 - x_k^j)\omega_k^j + (t_{\bar{o}} - t_k^j - \Delta_k^j)\} + A_k^j(1 - x_k^j) + \sqrt{A_k^j\Psi_k^jD_k^j}(t_{\bar{o}} - t_k^j)] - \sum_{j=1}^{\lvert J\rvert}(1 + t_{\bar{o}})h^j{*}\overline{V^j} - \sum_{o=1}^{\lvert O\rvert}B_o(1 - x_o) - \sum_{n\in\{E_n\}^+}^{\lvert N\rvert}A_n(1 - x_n)$
4	ω_n	$\omega_n = \sqrt{\frac{A_n}{E_n\Psi_n}}$

$$TC2(E_n) = 2\sum_{n=1}^{|N|}\sqrt{A_n\Psi_nE_n} - \sum_{n=1}^{|N|}\kappa_n\Delta t_nE_n \qquad (28)$$

Minimizing Eq. 27 under the constraint Eq. A2 constitutes a separate optimization problem that should be solved first to obtain the values of D_k^j and D_i. Equations 12 and 13 provide the cycle times ω_k^j and ω_i. Equation 18 provides the startup times t_k^j and t_i^{in} following the method used by Yi and Reklaitis(2003). Then, minimizing Eq. 28 under the constraint of Eq. 19 gives the values of E_n and t_n. Finally ω_n can be calculated using Eq. 14. Equation 19 is nonconvex and nonseparable, and, therefore, minimizing Eq. 28 under the constraint of Eq. 19 is a nontrivial task. Equation 19 can become a separable concave function, only when t_n and $t_{\bar{o}}$ are already known. Note that binary variables are required to resolve the term $\sum_{n\in\{E_n\}^+}^{|N|} A_n(1 - x_n) = \sum_{n=1}^{|N|} A_n(1 - x_n)z_n$, where $z_n = 1$ if $E_n > 0$ and $z_n = 0$ otherwise. Table 7 summarizes the procedure of the suboptimal approach.

The suboptimal approach suggested in this section is optimal if all the average material flow rates are determined by Eq. A2. This corresponds to the case in which the plant structure does not have any flexibility in regard to the material flow path. The suboptimal approach is also optimal if the level of financial

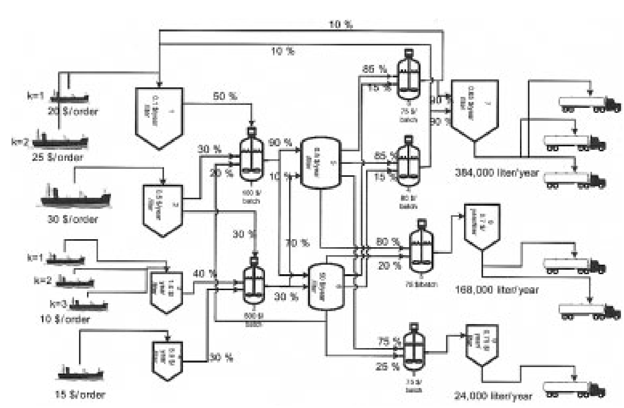

Figure 3. Example plant - input data.

Table 8. Financial Input and Output Data of Example

	(Input Data)									
	h^j (\$/L/year)	γ^j (\$/L/year)	S_m^j (\$/L/year)	Δt_m^j (year)	Δt_{k1}^j (year)	Δt_{k2}^j (year)	Δt_{k3}^j (year)	P_{k1}^j (\$/L)	P_{k2}^j (\$/L)	P_{k3}^j (\$/L)
J1	0.03	0.07			1/12	1/12		1004	1005	
J2	0.2	0.3			1/12			1006		
J3	0.3	0.7			1/12	1/12	1/12	1002	1002	1002
J4	2	3			1/12			1003		
J5	0.2	0.3								
J6	20	30								
J7	0.25	0.4	1006	1/12						
J8	0.3	0.4	1008	1/12						
J9	0.3	0.45	1010	1/12						

	(Input Data)			
	κ_n (\$/\$/year)	Δt_n (year)	a_n (\$/\$/year), x_n	A_n (\$/transaction)
N1	0.1	0.5	0	10
N2	0.07	1/12	0	10
η = 0.05 (\$/\$/year)				

	(Output Data)			
	E_n (\$/year)	$E_n\omega_n$ (\$)	ω_n (year)	t_n (year)
N1	82850	4021	0.04853	0.7
N2	0	0	0	0

investment is fixed, that is, E_n = constant. Although the solution identified by the suboptimal approach is not guaranteed to be the global optimum, the suboptimal solution should be very close to the global optimal solution because the optimal solution of objective function $TC1(D_k^j, D_i)$. in Eq. 27, with the constraint Eq. A2 tends to increase the value of the right-hand side of Eq. 19, and consequently to decrease the value of $TC2(E_n)$ in Eq. 28. Therefore, the suboptimal approach is suitable for manufacturing companies whose cash flows related to financial investments are usually small compared to those related to material flows.

Discussion with an Example Plant Design

We used the same plant design example in Yi and Reklaitis (2003) in which cash availability had been implicitly assumed unlimited. The plant structure is depicted in Figure 3. In this study, we included cash flows and financial transactions in the model by means of installing a cash storage unit. Additional input data are summarized in Table 8. The initial cash inventory was set to \$3,332,087 which was calculated by Eq. 26. For simplicity, the flow of dividends to stockholders was assumed to be continuous. The startup time of dividend payments to stockholders was set to 0.4 year, as obtained from Eq. 25. Sales tax was ignored because, from Eq. 19, the effect of sales tax on cash flow is negligible ($\varsigma[\hat{t}_m^j - t_m^j - \Delta t_m^j - (1 - \hat{x})\hat{\omega}] \approx 0$). First we considered the case in which no temporary financial investment occurred (CASE1). The suboptimal approach summarized in Table 7 was used sequentially. Table 9 compares the sizes of the processes and storage units calculated using the equations in this study and those of Yi and Reklaitis (2003). Notice that the optimal equipment sizes become smaller when cash flows are included in the calculations. In contrast, inclusion of cash flows has a negligible effect on the sizes of the batch production processes because the values of the additional terms in Eq. 16, ηh^j, are very small in this example. Figure 4 shows the optimal cash inventory profile for CASE1 as calculated using Eq. 4. The maximum cash inventory calculated by $*\bar{\bar{C}} = 2*\bar{C}$ and Eq. 21 was \$1,114,406, as identified from Figure 4. The term $\sum_{j=1}^{|J|} h^j \int_0^t V^j(t)dt$ was computed numerically by using Eq. A3. Note that a huge initial cash inventory

Table 9. Comparison of Plant Design Results

Process		Batch Size of Yi and Reklaitis (2003)	This Study
I1		113	113
I2		336	336
I3		239	239
I4		136	136
I5		171	171
I6		64	64
Storage	**Supplier**	**Order Size of Yi and Reklaitis (2003)**	**This Study**
J1	K1	272	177
J2	K2	117	76
J2	K1	570	371
J3	K1	283	185
J3	K2	73	48
J3	K3	73	48
J4	K1	304	204
Storage		**Storage Size of Yi and Reklaitis (2003)**	**This Study**
J1		483	347
J2		704	505
J3		563	415
J4		405	305
J5		840	840
J6		241	241
J7		337	337
J8		171	171
J9		64	64

is required to cover the initial cash shortage, which mostly originates from raw material purchases before the collection of accounts receivable. This initial cash requirement could be covered by a bank loan, but the source of the initial cash inventory is not considered in this study.

To demonstrate the case in which financial investments are made (CASE2), the startup time of stockholder dividends was delayed from 0.4 year used in the earlier case to 0.5 year. This delay was imposed to cause the buildup of a small amount of excess cash that could be used for financial investments N1 and N2, with the input data in Table 8. The startup times of N1 and N2 were set to 0.7 years. The separable concave minimization problem in the steps 1 and 3 of Table 7 was solved by the specially ordered sets (SOS) method, as explained in Appendix D. The separable nonlinear terms in the objective function and constraint were linearized by 5 equal intervals. The convergence gap was set to 0.0% with GAMS/CPLEX 7.0. The entire computation ran in less than 1 second on an ordinary personal computer. The output data is listed in Table 8. Figure 5 shows the cash inventory profiles obtained for CASE1 and CASE2. The difference between the cash inventory profiles for the two cases gradually diminishes as time progresses. The average dividend flow rate was increased from 479,569 \$/year for CASE1 to 483,506 \$/year for CASE2. Remember that the dividend payment was delayed by 0.1 year instead of such benefit increase. The maximum cash inventory level was slightly increased from \$1,114,406 for CASE1 to \$1,122,658 for CASE2 which meant that the quality of the cash inventory management was degraded by adding financial investments. The same result was obtained when we solved the second level nonlinear programming problem of the example with GAMS/CONOPT. In order to compute the derivatives of square roots near to zero, we put the lower bounds of average flow rates greater than 1e–5. Because the setup costs were relatively small, integer variables were not introduced by assuming that all flow rates were positive. The problem also converged within 1 second.

Conclusion

In this article, we have extended the common production optimization model to include simultaneous decision-making on production and financing. The optimal production plan in the presence of binding financial constraints differed from the plan generated under the assumption of unlimited cash availability. Inclusion of financial factors in the model led to a decrease in the optimal production lot and storage sizes. The objective function of the optimization was minimizing the opportunity costs of annualized capital investment and cash/material inventory minus stockholder benefits. Backlogging costs of the cash and material inventories and sequence dependent production setup costs were not considered in this study. The average flow rates of material and cash flows were calculated by solving separable concave minimization problems by using a piecewise linearization technique. Lot sizes and start-up times were determined by analytical equations. Despite the enlarged scope of the problem, the computational burden was light due to the use of mostly analytical results and the numerically easy subproblem structure, such as separable concave minimization. The batch-storage network used in this study is very general to cover most business activities, such as raw material procurement, transportation, labor, tax, as well as production and financial transaction. This study will contribute to preventing even a profitable company from being bankrupt because of bad management of operating cash flows, and lead to genuine enterprise optimization.

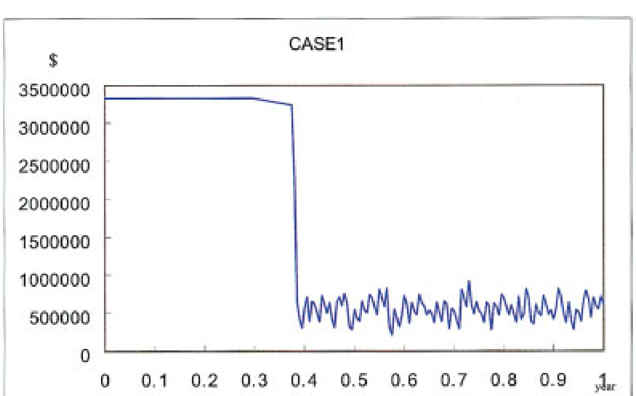

Figure 4. Cash inventory profile resulted from optimal design without financial investment (CASE1).

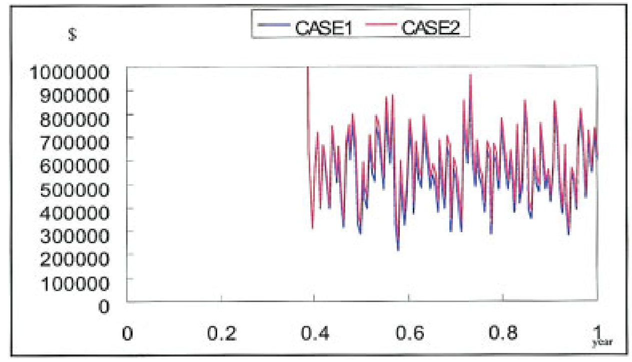

Figure 5. Cash Inventories with (CASE2) and without (CASE1) financial investment.

Acknowledgment

This work was supported by a grant No.(R01-2002-000-00007-0) from Korea Science & Engineering Foundation.

Notation

a_k^j: = annualized capital cost of raw material purchasing facility, \$/L/year
a_i: = annualized capital cost of unit i, \$/L/year
a_n: = annualized financial investment cost proportional to investment size, \$/\$/year
b^j: = annualized capital cost of storage facility, \$/L/year
A_k^j: = ordering cost of feedstock materials, \$/batch
A_i: = ordering cost of noncontinuous units, \$/batch
A_n: = ordering cost of financial investment, \$/transaction
B_k^j: = raw material order size, L/batch
B_i: = noncontinuous unit size, L/batch
B_m^j: = final product delivery size, L/batch
B_o: = lot size of dividend to stockholders, \$/lot
$C(0)$: = initial cash inventory, \$
$C(t)$: = cash inventory at present time t, \$
$\bar{C}$: = average level of cash inventory, \$
$\bar{\bar{C}}$: = upper level of cash inventory, \$
$\underline{C}$: = lower level of cash inventory, \$
$\bar{D}_k^j$: = average material flow of raw material supply, L/year

$\{D_k^j\}^+$: = set of index k that has positive value of D_k^j
D_m^j: = average material flow of customer demand, L/year
D_i = average material flow through noncontinuous units, L/year
$\{D_i\}^+$ = set of index i that has positive value of D_i
E_n = average cash flow rate of financial investment n, \$/year
$\{E_n\}^+$ = The set of index n that has positive value of E_n
$\hat{E}$ = average cash flow rate of sales tax, \$/year
^j_i = feedstock composition of unit i
g_i^j = product yield of unit i
H^j = annual inventory holding cost, \$/L/year
h^j = annual inventory operating cost, \$/L/year
I = noncontinuous process set
J = storage set
$K(j)$ = raw material supplier set for storage, j
$M(j)$ = consumer set for storage, j
P_k^j = price of raw material j from supplier k, \$/L
PSW = first type of periodic square wave flow defined by Eq. 7
PSW' = second type of periodic square wave flow defined by Eq. 8
$\overline{PSW}$ = average level of the first type of periodic square wave flow
$\overline{PSW'}$ = average level of the second type of periodic square wave flow
$\overline{\overline{PSW}}$ = upper bound of the first type of periodic square wave flow
$\overline{\overline{PSW'}}$ = upper bound of the second type of periodic square wave flow
$\underline{\underline{PSW}}$ = lower bound of the first type of periodic square wave flow
$\underline{\underline{PSW'}}$ = lower bound of the second type of periodic square wave flow
S_m^j = sales price of finished products, \$/L
t_m^j = startup time of customer demand, year
t_i^{in} = startup time of feedstock feeding to noncontinuous unit i, year
t_i^{out} = startup time of product discharging from noncontinuous unit i, year
t_k^j = startup time of raw material purchasing, year
t_c = minimum of the collection times of account receivables, year
t_n = startup time of financial investment n, year
$t_{\bar{o}}$ = startup time of dividend to stockholders, year
$\hat{t}_m^j$ = startup time of sales tax, year
Δt_k^j = disbursement drifting time of account payables, year
Δt_m^j = collection drifting time of account receivables, year
Δt_n = financial investment period, year
$\overline{\overline{V^k}}$ = upper bound of inventory hold-up, L
$\underline{\underline{V^j}}$ = lower bound of inventory hold-up, L
$V^j(t)$ = inventory holdup, L
$V^j(0)$ = initial inventory holdup, L
$\overline{V^j}$ = time averaged inventory holdup, L
x_k^j = storage operation time fraction of purchasing raw materials
x_i^{in} = storage operation time fraction of feeding to noncontinuous unit i
x_i^{out} = storage operation time fraction of discharging from noncontinuous unit i
x_j^m = storage operation time fraction of customer demand
x_n = storage operation time fraction of financial investment
x_o = storage operation time fraction of dividend to stockholders
$\hat{x}$ = storage operation time fraction of sales tax

Greek letters

ς = Sales tax rate, \$/\$
γ^j = opportunity cost of inventory holding, \$/L/year
η = opportunity cost of cash holding, \$/\$/year
κ_n = interest rate of financial investment n, \$/\$/year
ω_m^j = cycle time of customer demand, year
ω_k^j = cycle time of raw material purchasing, year
ω_i = cycle time of noncontinuous units, year
ω_n = cycle time of financial investment n, year
ω_o = cycle time of dividend to stockholders, year
$\hat{\omega}$ = cycle time of sales tax, year
Ψ_i = aggregated cost for process i defined by Eq. 29, \$/L/year
Ψ_k^j = aggregated cost for raw material purchase defined by Eq. 30, \$/L/year
Ψn = aggregated cost for financial investment n defined by Eq. 31, \$/\$/year

Subscript

i = noncontinuous unit index
l = similar process group index
k = raw material vendors
m = finished product customers
n = financial investments
o = stockholders

Superscript

j = storage index

Special functions

$int[.]$ = truncation function to make integer
$res[.]$ = positive residual function to be truncated
$|X|$ = Number of elements in set X

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Appendix A: The Equations Derived by Yi and Reklaitis (2003)

From the fact that one production cycle in a process is composed of the feedstock feeding time ($x_i^{in}\omega_i$), processing time ($(1 - x_i^{in} - x_i^{out})\omega_i$), and product discharging time ($x_i^{out}\omega_i$), there exists the following timing relationship between the startup time of the feedstock streams and the startup time of the product streams

$$t_i^{out} = t_i^{in} + \omega_i(1 - x_i^{out}) \tag{A1}$$

The overall material balance associated with the storage results in the following relationships

$$\sum_{i=1}^{|I|} g_i^j D_i + \sum_{k=1}^{|K(j)|} D_k^j = \sum_{i=1}^{|I|} f_i^j D_i + \sum_{m=1}^{|M(j)|} D_m^j \tag{A2}$$

A storage unit is connected to the incoming flows from suppliers and processes, and the outgoing flows to consumers and processes. The resulting inventory holdup function for a storage unit is

$$V^j(t) = V^j(0) + \sum_{k=1}^{|K(j)|} B_k^j \left[\text{int}\left[\frac{t - t_k^j}{\omega_k^j}\right] + \min\left\{1, \frac{1}{x_k^j} res\left[\frac{t - t_k^j}{\omega_k^j}\right]\right\}\right]$$

$$+\sum_{i=1}^{|I|}(g_i^jB_i)\left[\text{int}\left[\frac{t-t_i^{out}}{\omega_i}\right]+\min\left\{1,\frac{1}{x_i^{out}}res\left[\frac{t-t_i^{out}}{\omega_i}\right]\right\}\right]$$
$$-\sum_{m=1}^{|M(j)|}B_m^j\left[\text{int}\left[\frac{t-t_m^j}{\omega_m^j}\right]+\min\left\{1,\frac{1}{x_m^j}res\left[\frac{t-t_m^j}{\omega_m^j}\right]\right\}\right]$$
$$-\sum_{i=1}^{|I|}(f_i^jB_i)\left[\text{int}\left[\frac{t-t_i^{in}}{\omega_i}\right]+\min\left\{1,\frac{1}{x_i^{in}}res\left[\frac{t-t_i^{in}}{\omega_i}\right]\right\}\right] \quad \text{(A3)}$$

The upper bound of the inventory holdup, the lower bound of the inventory holdup and the average inventory holdup are calculated by using the properties of the flow accumulation function (Yi and Reklaitis 2002)

$$\overline{\overline{V^j}}=V^j(0)+\sum_{k=1}^{|K(j)|}(1-x_k^j)D_k^j\omega_k^j-\sum_{k=1}^{|K(j)|}D_k^jt_k^j+\sum_{i=1}^{|I|}(1$$
$$-x_i^{out})g_i^jD_i\omega_i-\sum_{i=1}^{|I|}g_i^jD_it_i^{out}+\sum_{i=1}^{|I|}f_i^jD_it_i^{in}+\sum_{m=1}^{|M(j)|}D_m^jt_m^j \quad \text{(A4)}$$

$$\underline{\underline{V^j}}=V^j(0)-\sum_{k=1}^{|K(j)|}D_k^jt_k^j-\sum_{i=1}^{|I|}g_i^jD_it_i^{out}-\sum_{i=1}^{|I|}(1-x_i^{in})f_i^jD_i\omega_i$$
$$+\sum_{i=1}^{|I|}f_i^jD_it_i^{in}-\sum_{m=1}^{|M(j)|}(1-x_m^{in})D_m^j\omega_m^j+\sum_{m=1}^{|M(j)|}D_m^jt_m^j \quad \text{(A5)}$$

$$\overline{V^j}=V^j(0)+\sum_{k=1}^{|K(j)|}\frac{(1-x_k^j)}{2}D_k^j\omega_k^j-\sum_{k=1}^{|K(j)|}D_k^jt_k^j$$
$$+\sum_{i=1}^{|I|}\frac{(1-x_i^{out})}{2}g_i^jD_i\omega_i-\sum_{i=1}^{|I|}g_i^jD_it_i^{out}-\sum_{i=1}^{|I|}\frac{(1-x_i^{in})}{2}f_i^jD_i\omega_i$$
$$+\sum_{i=1}^{|I|}f_i^jD_it_i^{in}-\sum_{m=1}^{|M(j)|}\frac{(1-x_m^j)}{2}D_m^j\omega_m^j+\sum_{m=1}^{|M(j)|}D_m^jt_m^j \quad \text{(A6)}$$

Equation A4 will be used to predict storage size; Eq. A5 will be used to implement the no depletion constraint; and Eq. A6 will be used to calculate the inventory holding cost.

Appendix B: Proof that the Kuhn-Tucker Conditions of the Two-Level Parametric Optimization Problem are the Same as the Original Optimization Problem if the Constraints become Equalities

We will consider only one variable, one parameter and one constraint but the formalism can be readily expanded to the case of multiple variables, parameters and constraints. Consider the following optimization problem

$$\min_{x,y} f(x, y) \textit{ subject to } g(x, y) \geq 0 \quad \text{(B1)}$$

The Kuhn-Tucker conditions for Eq. B1 are

$$\frac{\partial f(x,y)}{\partial x}-\lambda\frac{\partial g(x,y)}{\partial x}=0$$
$$\frac{\partial f(x,y)}{\partial y}-\lambda\frac{\partial g(x,y)}{\partial y}=0$$
$$\lambda g(x,y)=0$$
$$\lambda \geq 0 \quad \text{(B2)}$$

Now, consider the optimization problem Eq. B1 separated into a two-level parametric problem. At first, suppose that y is a fixed parameter. Under this condition, the first level optimization problem is

$$\min_{x} f(x, y) \textit{ subject to } g(x, y) \geq 0,\ \forall y \quad \text{(B3)}$$

The Kuhn-Tucker conditions for Eq. B3 are

$$\frac{\partial f(x,y)}{\partial x}-\lambda\frac{\partial g(x,y)}{\partial x}=0$$
$$\lambda g(x,y)=0$$
$$\lambda \geq 0 \quad \text{(B4)}$$

Let $\bar{x}(y)$ and $\bar{\lambda}(y)$ be the solution of Eq. B4. If we suppose that $g(\bar{x}(y), y) = 0$. Then, the second level optimization is

$$\min_{y} f(\bar{x}(y), y) \textit{ subject to } g(\bar{x}(y), y) = 0 \quad \text{(B5)}$$

and the Kuhn-Tucker condition is

$$\frac{df(\bar{x}(y),y)}{dy}-\lambda'\frac{dg(\bar{x}(y),y)}{dy}=0 \quad \text{(B6)}$$

Note that although Eq. B6 includes total differentials, it can be reduced to partial differentials by applying the chain rule to Eq. B6 with $\lambda' = \bar{\lambda}(y)$.

$$\frac{df}{dy}-\lambda'\frac{dg}{dy}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial f}{\partial y}-\lambda'\left[\frac{\partial g}{\partial x}\frac{\partial x}{\partial y}+\frac{\partial g}{\partial y}\right]=\left[\frac{\partial f}{\partial x}\right.$$
$$\left.-\lambda'\frac{\partial g}{\partial x}\right]\frac{\partial x}{\partial y}+\frac{\partial f}{\partial y}-\lambda'\frac{\partial g}{\partial y}=\frac{\partial f}{\partial y}-\lambda'\frac{\partial g}{\partial y}=0 \quad \text{(B7)}$$

Equations B4 and B7 coincide with Eq. B2 under the assumption of $g(\bar{x}(y), y) = 0$. This proves our statements.

Appendix C: The Solution of the Kuhn-Tucker Conditions of the First Problem

Because the constraints are $\underline{\underline{V^j}} \geq 0$ and $\underline{\underline{C}} \geq 0$, the Lagrangian is

$$L = TC - \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j \underline{\underline{V}}^j - \underline{\underline{\lambda}}\,\underline{\underline{C}} \tag{C1}$$

where $\underline{\underline{\lambda}}^j$ and $\underline{\underline{\lambda}}$ are nonnegative Lagrange multipliers. For simplicity, we do not explicitly include the nonnegativity constraints of the design variables. The Kuhn-Tucker conditions of Eq. C1 are

$$\frac{\partial L}{\partial t_k^j} = \left[(h^j + \gamma^j)\frac{\partial \overline{V^j}}{\partial t_k^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial t_k^j} \right] - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial t_k^j} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial t_k^j} + \eta \frac{\partial \bar{C}}{\partial t_k^j}$$
$$= -(h^j + \gamma^j + b^j) D_k^j + \underline{\underline{\lambda}}^j D_k^j - \underline{\underline{\lambda}} \left[P_k^j D_k^j + \frac{A_k^j}{\omega_k^j} + \frac{\partial}{\partial t_k^k}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] + \eta \left[P_k^j D_k^j + \frac{A_k^j}{\omega_k^j} + \frac{\partial}{\partial t_k^k}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C2}$$

$$\frac{\partial L}{\partial \omega_k^j} = \left[-\frac{A_k^j}{(\omega_k^j)^2} + a_k^j D_k^j + (h^j + \gamma^j)\frac{\partial \overline{V^j}}{\partial \omega_k^j} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \omega_k^j} \right] - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \omega_k^j} + \eta \frac{\partial \bar{C}}{\partial \omega_k^j} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial \omega_k^j} = \left[-\frac{A_k^j}{(\omega_k^j)^2} + \left[\left(\frac{h^j + \gamma^j}{2} + b^j \right)(1 - x_k^j) + a_k^j \right] D_k^j \right] - \eta \left[\frac{(P_k^j)(1 - x_k^j)}{2} D_k^j + \frac{A_k^j t_k^j}{(\omega_k^j)^2} + \frac{\partial}{\partial \omega_k^j}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] + \underline{\underline{\lambda}} \left[(0.5h^j + P_k^j)(1 - x_k^j) D_k^j + \frac{A_k^j t_k^j}{(\omega_k^j)^2} + \frac{\partial}{\partial \omega_k^j}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C3}$$

$$\frac{\partial L}{\partial t_i^{in}} = \sum_{j=1}^{|J|} \left[(h^j + \gamma^j)\frac{\partial \bar{V}}{\partial t_i^{in}} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial t_i^{in}} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial t_i^{in}} \right] + \eta \frac{\partial \bar{C}}{\partial t_i^{in}} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial t_i^{in}} = \sum_{j=1}^{|J|} (h^j + \gamma^j + b^j)(f_i^j - g_i^j) D_i - \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j (f_i^j - g_i^j) D_i + \eta \left[\frac{A_i}{\omega_i} + \frac{\partial}{\partial t_i^{in}}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] - \underline{\underline{\lambda}} \left[\frac{A_i}{\omega_i} + \frac{\partial}{\partial t_i^{in}}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C4}$$

$$\frac{\partial L}{\partial \omega_i} = -\frac{A_i}{(\omega_i)^2} + a_i D_i + \sum_{j=1}^{|J|} \left[(h^j + \gamma^j)\frac{\partial \overline{V^j}}{\partial \omega_i} + b^j \frac{\partial \overline{\overline{V^j}}}{\partial \omega_i} - \underline{\underline{\lambda}}^j \frac{\partial \underline{\underline{V^j}}}{\partial \omega_i} \right] + \eta \frac{\partial \bar{C}}{\partial \omega_i} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial \omega_i} = -\frac{A_i}{(\omega_i)^2} + a_i D_i - \sum_{j=1}^{|J|} \frac{h^j + \gamma^j}{2} [(1 - x_i^{in}) f_i^j + (1 - x_i^{out}) g_i^j] D_i + \sum_{j=1}^{|J|} \underline{\underline{\lambda}}^j [(1 - x_i^{in}) f_i^j + (1 - x_i^{out}) g_i^j] D_i - \eta \left[\frac{A_i t_i^{in}}{(\omega_i)^2} + \frac{\partial}{\partial \omega_i}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] + \underline{\underline{\lambda}} \left[\sum_{j=1}^{|J|} h^j \left[\frac{(1 - x_i^{in})}{2} f_i^j D_i + \frac{(1 - x_i^{out})}{2} g_i^j D_i \right] + \frac{A_i t_i^{in}}{(\lambda_i)^2} + \frac{\partial}{\partial \omega_i}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C5}$$

$$\frac{\partial L}{\partial t_n} = \eta \frac{\partial \bar{C}}{\partial t_n} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial t_n} = \eta \left[-\Delta t_n \kappa_n E_n + \frac{A_n}{\omega_n} + \frac{\partial}{\partial t_n}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] - \underline{\underline{\lambda}} \left[-\Delta t_n \kappa_n E_n + \frac{A_n}{\omega_n} + \frac{\partial}{\partial t_n}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C6}$$

$$\frac{\partial L}{\partial \omega_n} = -\frac{A_n}{(\omega_n)^2} + a_n E_n + \eta \frac{\partial \bar{C}}{\partial \omega_n} - \underline{\underline{\lambda}} \frac{\partial \underline{\underline{C}}}{\partial \omega_n} = -\frac{A_n}{(\omega_n)^2} + a_n E_n + \eta \left[0.5(1 - x_n) E_n \kappa_n \Delta t_n - \frac{A_n t_n}{(\omega_n)^2} + \frac{\partial}{\partial \omega_n}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] - \underline{\underline{\lambda}} \left[-(1 - x_n) E_n - \frac{A_n t_n}{(\omega_n)^2} + \frac{\partial}{\partial \omega_n}\left(t_{\bar{o}} \sum_{o=1}^{|O|} \frac{B_o}{\omega_o} \right) \right] = 0 \tag{C7}$$

$$\underline{\underline{\lambda}}^j \underline{\underline{V}}^j = 0 \quad \text{and} \quad \underline{\underline{\lambda}}\,\underline{\underline{C}} = 0 \tag{C8}$$

The partial derivatives of the important functions in the Lagrangian are summarized in Table 5. Solving Eqs. C2, C4 and C6 gives the values of multipliers

$$\underline{\underline{\lambda}}^j = h^j + \gamma^j + b^j, \quad \underline{\underline{\lambda}} = \eta \tag{C9}$$

Solving Eqs. C3, C5 and C7 with Eq. C9 gives the optimal values of the cycle times in the main text.

Appendix D: Piecewise Linear Approximation of Separable Function by Using Specially Ordered Sets

The following equations show how to piecewisely linearize $\sqrt{A_i \Psi_i D_i}$ with respect to as a suitable formulation for specially ordered sets

$$\sum_n \lambda_i^n = 1$$

$$D_i = \sum_n \lambda_i^n D_i^n$$

$$\sqrt{A_i \Psi_i D_i} = \sum_n \lambda_i^n \sqrt{A_i \Psi_i D_i^n}$$

$$D_i^n = D_i^{\min} + \frac{n}{N}(D_i^{\max} - D_i^{\min})$$

$$(n = 0, 1, 2, \ldots, N) \quad \text{(D1)}$$

where D_i and λ_i^n are nonnegative variables and the other variables are treated as parameters. λ_i^n is specially ordered sets that has the property of $\lambda_i^{n'} + \lambda_i^{n'+1} = 1$ and $\lambda_i^0, \lambda_i^1, \ldots, \lambda_i^{n'-1}, \lambda_i^{n'+2}, \ldots, \lambda_i^N = 0$. GAMS/CPLEX has the special ability to deal with specially ordered sets by declaring SOS2 VARIABLES (Brooke et. al., 1998). $N = 5$ is sufficient to approximate a square root function.

Manuscript received Sept. 8, 2003, revision received Feb. 19, 2004, and final revision received Apr. 19, 2004.